NUMERICAL SOLUTION OF THE PROBLEM OF THE MOTION OF A FLUID IN A RECTANGULAR HOLE

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We shall consider the motion of a viscous incompressible fluid in a flat-bottomed rectangular hole created by the motion of a plane (Fig. 1). Introduction of the stream function reduces the Navier-Stokes equations, as usual, to the equations

$$\frac{\partial \Delta\psi}{\partial t} + R\left(\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right) = \Delta(\Delta\psi) \tag{1}$$

$$\left(u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x}\right),$$

where u and v are dimensionless velocity components, x, y, and t are dimensionless arguments, and R is the Reynolds number.

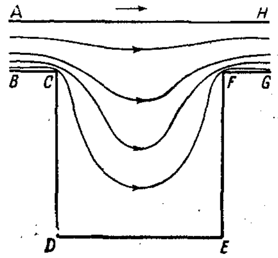

Fig. 1. R = 0, k_1 = 2, t = 0.2.

It is assumed that the plane AH moves according to the law

$$u = 1 - e^{-k_1 t}\quad (k_1 = \text{const}).$$

The dimensional velocities v_ξ and v_η and the arguments τ, ξ, η are related by the dimensional equations

$$u = \frac{v_\xi}{U},\quad v = \frac{v_\eta}{U},\quad y = \frac{\eta}{h},\quad x = \frac{\xi}{h},\quad t = \frac{\tau\nu}{h^2}\qquad (U = \text{const}).$$

We solve equation (1) for the following initial and boundary conditions

$$\psi = {}^1/_2\,(1 - e^{-k_1 t}) \text{ on } BCDEFG$$

$$\partial\psi / \partial y = 0 \text{ on } BC,\ DE \text{ and } FG \quad \partial\psi / \partial x = 0 \text{ on } AB,\ CD,\ EF \text{ and GH}$$

$$\psi = {}^1/_2\,(1 - e^{-k_1 t})\, y^2 \text{ on } AB \text{ and } GH$$

$$\psi = 0,\ \partial\psi / \partial y = 1 - e^{-k_1 t} \text{ on } AH \qquad \psi = 0 \text{ for } t = 0.$$

Thus we assume that

$$u = (1 - e^{-k_1 t})\, y,\ v = 0 \text{ on the lines } AB \text{ and } GH$$

We reduce equation (1) to the system

$$\frac{\partial\varphi}{\partial t} + R\left(\frac{\partial\psi}{\partial y}\frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial y}\right) = \Delta\varphi,\quad \Delta\psi = \varphi \tag{2}$$

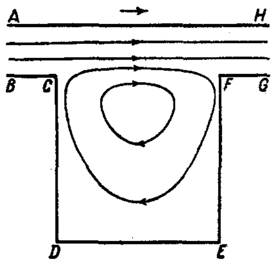

Fig. 2. R = 0, k_1 = 2, t = 2.

We substitute finite-difference relations for the derivatives in system (2), and introduce an output-time grid (t_n, x_i, y_k), where t_n = = $n\Delta t$, $x_i = i\Delta x$, $y_k = k\Delta y$ (n = 0, 1, 2,..., i = 0, 1, 2,..., I; k = 0, 1, 2,..., K).

The resulting difference system is

$$\begin{gathered}\varphi_{i,k}^{\;n+1} = \varphi_{i,k}^{\;n} + \Delta t\left\{\frac{\varphi_{i,k+1}^{\;n} + \varphi_{i,k-1}^{\;n}}{(\Delta y)^2} + \frac{\varphi_{i+1,k}^{\;n} + \varphi_{i-1,k}^{\;n}}{(\Delta x)^2} - \right.\\ -\frac{2\,[(\Delta x)^2 + (\Delta y)^2]\,\varphi_{i,k}^{\;n}}{(\Delta x)^2(\Delta y)^2} - R\left(\frac{\psi_{i,k+1}^{\;n} - \psi_{i,k-1}^{\;n}}{2\Delta y}\,\frac{\varphi_{i+1,k}^{\;n} - \varphi_{i-1,k}^{\;n}}{2\Delta x} - \right.\\ \left.\left. - \frac{\psi_{i-1,k}^{\;n} - \psi_{i-1,k}^{\;n}}{2\Delta x}\,\frac{\varphi_{i,k+1}^{\;n} - \varphi_{i,k-1}^{\;n}}{2\Delta y}\right)\right\},\\ (\psi_{i+1,k}^{\;n+1} + \psi_{i-1,k}^{\;n+1})(\Delta y)^2 + (\psi_{i,k+1}^{\;n+1} + \psi_{i,k-1}^{\;n+1})(\Delta x)^2 - \\ - 2\,[(\Delta x)^2 + (\Delta y)^2]\,\psi_{i,k}^{\;n+1} = \varphi_{i,k}^{\;n+1}(\Delta x)^2(\Delta y)^2.\end{gathered} \tag{3}$$

We assume that at a certain moment t_n, the values of the quantities $\varphi_{i,k}^{n}$ and $\psi_{i,k}^{n}$ are known at each point of the grid region. With the aid of formula (3) we determine the values of the quantities $\varphi_{i,k}^{n+1}$ at the internal points of the region. Then we determine the values of the quantity $\psi_{i,k}^{n+1}$ by solving the difference analog (4) of the Poisson equation. The boundary values of the quantities $\varphi_{i,k}^{n+1}$ are determined from formulas first derived by A. Thom [1]. For the wall y = 0, this formula has the form

$$\varphi_{i,0}^{\;n+1} = \frac{2\,[\psi_{i,1}^{\;n+1} - (1 - e^{-k_1 t_{n+1}})\,\Delta y]}{(\Delta y)^2}.$$

Analogous formulas can be obtained for the other boundaries by expanding the quantity $\psi_{i,k}^{n+1}$ at a point near the boundary in a Taylor series and using the second equation of system (2) at the boundary. The calculations were performed for the following cases (all dimensions are referred to the channel width AB):

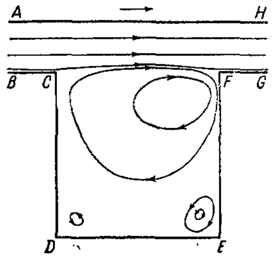

Fig. 3. R = 500, k_1 = 2, t = 1.5.

1. Motion of a fluid in a square cavity AB = 1, BC = FG = 1, CD = = DE = 3 1/3 (Fig. 1). a) R = 0 (neglecting inertial terms). A flow pattern symmetrical with respect to the symmetry line of the cavity can be observed at any moment, but the flow pattern itself undergoes substantial changes with time. Streamlines that correspond to the flow at the various moments of time are given in Figs. 1 and 2.

It can be seen that at the initial moments of time the streamlines are not closed and there is an exchange of fluid over the entire volume of the cavity with the external flow (Fig. 1). With increasing time, a region of reverse flow appears at the cavity walls, which gradually extends over the entire cavity (Fig. 2).

b) R = 100. The motion at the initial moments, for R = 100, is similar to that at R = 0. However, the onset of reverse currents takes place primarily at the left wall when the moving plane moves to the right. The flow develops nonsymmetrically and the core of the secondary flow is displaced to the right.

c) R = 500. The flow pattern scarcely differs qualitatively from that at R = 100. A corner vortex can be seen in the developed flow (Fig. 3).

2. Motion of a fluid in a deep cavity AB = 1, BC = FG = 1, CD = = 6 2/3, DE = 3 1/3. a) R = 100. During the initial moments (up to t = 0.2 for k = 2) all the fluid moves in one direction (as in Fig. 1 for a square cavity). Two regions of reverse flow gradually develop, forming the steady-state pattern (Fig. 4).

b) R = 500. The flow develops like that at R = 100, but the asymmetry of the flow is greater. The region of bottom flow for R = 500 is greater than for R = 100.

3. Motion of fluid in a shallow cavity AB = 1, BC = FC = 1, CD = = 1 2/3, DE = 3 1/3. a) R = 100. The flow pattern is similar to that obtained for a square cavity. The core of the secondary flow is displaced in the direction of the motion of the moving plane.

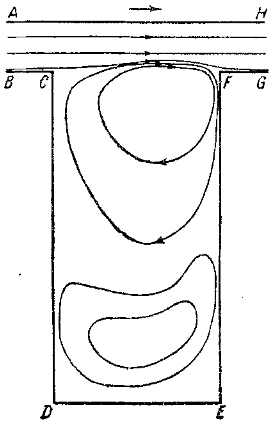

Fig. 4. R = 100, k_1 = 2, t = 2.

b) R = 500. The core of the secondary flow is displaced still further, while an additional closed-flow region appears in the opposite corner. The flow rates at the wall CD are very low and the motion approaches stagnation (Fig. 5).

Hydrodynamically, the solutions of all the cases examined exhibit certain common features. The secondary flows appear earlier (in terms of dimensionless time) at large Reynolds numbers. The dimensionless rate of flow along the line CF decreases with increasing Reynolds number: thus, the maximum values u_* of the velocity $u = \partial\psi/\partial y$ on the straight line connecting the points C and F (for a square cavity) are 0.20, 0.17, and 0.13 for R = 0, 100, and 500, respectively.

The velocity variation also exhibits the same nature in other cases.

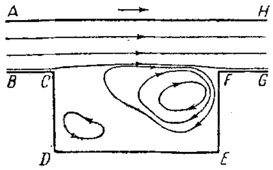

Fig. 5. R = 500, k_1 = 2, t = 1.5.

Calculations were performed for $\Delta x = 0.1$, $\Delta y = 0.1$; $\Delta x = 0.1$, $\Delta y = 0.05$; and $\Delta x = 0.05$, $\Delta y = 0.05$. This did not produce a change in the flow pattern, but the secondary vortex in the corner CDE in Fig. 3 could be detected only with the aid of a fine grid. The time interval Δt was varied in the calculations to assure stability at increasing velocities. The final choice for a square cavity was

$$\Delta t = 0.25 \cdot 10^{-2} \text{ for } R = 0,$$

$$\Delta t = 0.625 \cdot 10^{-3} \text{ for } R = 100,$$

$$\Delta t = 0.625 \cdot 10^{-4} \text{ for } R = 500.$$

The results indicate that in using numerical methods, it is possible to reveal some interesting features of viscous flows. All the calculations were performed on a BESM-2 computer at the Computer Center of the Leningrad Department of the Mathematical Institute of the Academy of Sciences, USSR.

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